Near resonance homogenization of split-ring metamaterials

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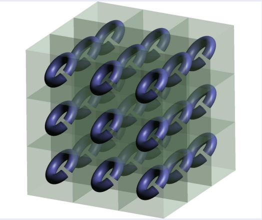
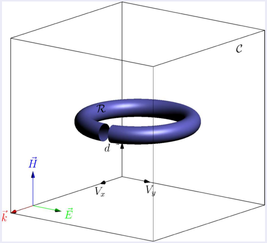
Introduction

- A split-ring array interacts with a planar wave $(\vec{E}, \vec{H}, \vec{k})$.
- Long wavelength limit $a \ll \lambda$: the fields $\vec{E}$ and $\vec{H}$ are considered uniform over the metamaterial.
- We wish to determine the effective permeability of this SRR based metamaterial.





Floquet-Bloch theorem

Theorem

A function $\varphi(x) \in L^2(\mathbb{R}^3, \mathbb{C})$ can be decomposed in the dual cell $\mathcal{C}^*$:

$$\varphi(x) = \int_{\mathcal{C}^*} e^{i\kappa\cdot x} \hat{\varphi}_\kappa(x) d\kappa \tag{1}$$

where Bloch amplitudes $\hat{\varphi}_\kappa(x)$ are C-periodic and represented by :

$$\hat{\varphi}_\kappa(x) = \sum_{n\in\mathbb{Z}^3} \check{\varphi}(\kappa + \kappa_n) e^{i\kappa_n\cdot(x+x_n)} = \frac{|\mathcal{C}|}{(2\pi)^3} \sum_{n\in\mathbb{Z}^3} \varphi(x + x_n) e^{-i\kappa\cdot(x+x_n)} \tag{2}$$

where $\check{\varphi}$ is the Fourier transform and κ the wave vector.

The dual cell $\mathcal{C}^*$

Generated by the vectors W^j, the duals of V_i. We have : $V_i W^j = 2\pi\delta_{ij}$

Homogenization by Floquet-Bloch decomposition

The derivative operator are changed

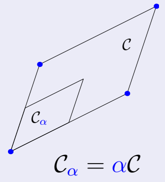
curl $= (\text{curl} + i\kappa\times)$ and div $= (\text{div} + i\kappa\cdot)$

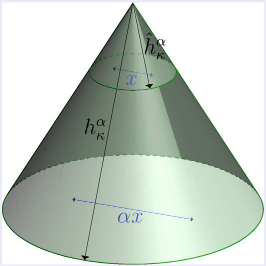
Maxwell equations (problem $\mathcal{P}_\kappa$)

$$
\begin{gathered}
-i\omega\hat{d}_\kappa + (\text{curl} + i\kappa\times)\hat{h}_\kappa = \hat{j}_\kappa \\
i\omega\hat{b}_\kappa + (\text{curl} + i\kappa\times)\hat{e}_\kappa = 0 \\
(\text{div} + i\kappa\cdot)\hat{d}_\kappa = \hat{q}_\kappa \quad (\text{div} + i\kappa\cdot)\hat{b}_\kappa = 0 \\
\hat{b}_\kappa = \mu\hat{h}_\kappa \quad \hat{d}_\kappa = \epsilon\hat{e}_\kappa
\end{gathered}
$$

Homogenization principle

Embedding the problem $\mathcal{P}_\kappa$ in a family of problems $\mathcal{P}_\kappa^\alpha$, and let $\alpha \to 0$. A scaling operation is necessary $h_\kappa^\alpha(\alpha x) = \hat{h}_\kappa^\alpha(\alpha x)$ to "pull back" all the magnetic field to the common domain $\mathcal{C}$.





The second small parameter : the split's width d

$\alpha \to 0$: static effective parameters

$$\begin{gathered} -i\omega \hat{d}_\kappa + i\kappa \times \hat{h}_\kappa = \alpha \hat{j}_\kappa \\ i\omega \hat{b}_\kappa + i\kappa \times \hat{e}_\kappa = 0 \\ \mathrm{div}\hat{b}_\kappa = 0 \quad \mathrm{curl}\hat{h}_\kappa = 0 \\ \hat{b}_\kappa = \mu_{eff}\hat{h}_\kappa \quad \hat{d}_\kappa = \epsilon_{eff}\hat{e}_\kappa \end{gathered} \tag{3}$$

Disappointing ! The obtained effective parameters are the static ones.

How to maintain the resonance ?

The resonance frequency $LC \sim \omega^{-2}$ must be a constant of all the problems $\mathcal{P}_\kappa^\alpha$ in order to be preserved at the limit. When the reference cell is shrank, L and C scale like $\sim \alpha$. The capacitance should scale as $\sim \alpha^{-1}$ to maintain the resonance. The corresponding slit's width is in $\sim \alpha^3$.

The slit's width d is considered as a second small parameter. It should tend to 0 faster than the period : $d \ll a \ll \lambda$.

To the closed ring

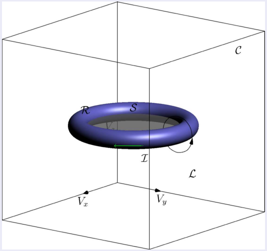
The weak formulation in h:

$$\alpha^3 \int_{\mathcal{A}} i\omega h^\alpha \cdot h' + \alpha \int_{\mathcal{A}} \frac{1}{i\omega\epsilon}((\mathrm{curl} + i\alpha\kappa)h^\alpha - \alpha j)(\mathrm{curl} + i\alpha\kappa)h' + \alpha^3 \int_{\Sigma} \frac{d}{i\omega\epsilon}(\vec{n}\cdot(\mathrm{curl} + i\alpha\kappa)h^\alpha)\cdot(\vec{n}\cdot(\mathrm{curl} + i\alpha\kappa)h') \quad (4)$$

The ring is closed

The ring now bears a capacitive ($C = \epsilon|\Sigma|/d$) layer Σ. As curl $h = 0$, a multivalued magnetic potential $\vec{h} = \vec{\nabla}(\varphi)$ exists. The magnetic potentiel jump is :

$$\mathcal{I} = \int_{\mathcal{L}} h = \int_{\mathcal{L}} \nabla\varphi = \varphi_{s+} - \varphi_{s-} = [\varphi] \quad (5)$$



The weak formulation

$$\int_{\mathcal{A}} \mu \nabla \varphi \cdot \nabla \varphi' + \int_{\partial \mathcal{R}} \frac{1-i}{\sigma \omega \delta} \nabla_S \varphi \cdot \nabla_S \varphi' - \frac{1}{C\omega^2} [\varphi][\varphi'] = \int_{\mathcal{A}} \mu \vec{H} \cdot \nabla \varphi' \tag{6}$$

Several constrains

$$\begin{aligned} \varphi_{x=a} - \varphi_{x=0} &= C_x \\ \varphi_{y=a} - \varphi_{y=a} &= C_y \\ \varphi_{z=a} - \varphi_{z=a} &= C_z \\ \varphi_{s+} - \varphi_{s-} &= \mathcal{I} \end{aligned}$$

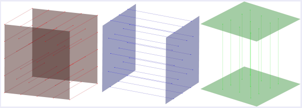
Choosing $\varphi' = \bar{\varphi}$, we obtain the effective permeability :

$$|\mathcal{C}| H \bar{\mu}_{eff} \bar{H} = \int_{\mathcal{A}} \mu |\nabla \varphi|^2 + \int_{\partial \mathcal{R}} \frac{1-i}{\sigma \omega \delta} |\nabla_S \varphi|^2 - \frac{1}{C\omega^2} \mathcal{I}^2 \tag{7}$$

The periodicity detection and the cutting surface modelization

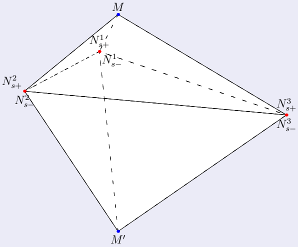
The periodicity

Each node on a face of the unit cell is identified with a node on the opposite face.

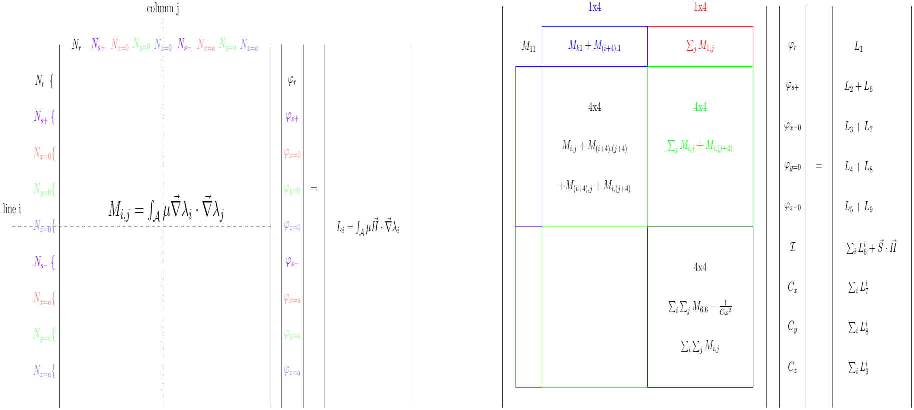


Modeling $[\varphi] = \mathcal{I}$

The nodes on the cutting surface are doubled and the connected elements to the cutting surface are transformed.

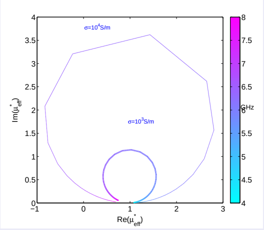


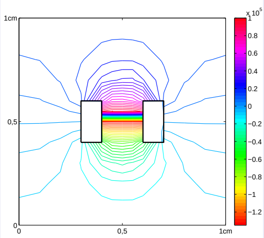
The stiffness matrix transformation



Permeability and isopotentials

- The permeability and the current tend to infinity if the ring is very conductive.
- Low conductivity means no negative permeability.
- The major part of the magnetic field flows through the cutting surface.
- At high and low frequencies, the isopotentials correspond to a diamagnetic material.



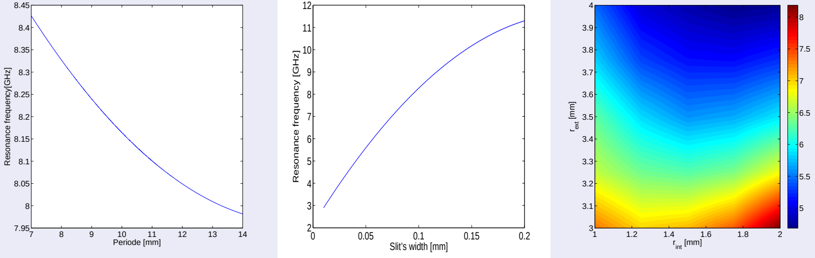


Influence of the ring's dimensions

a $\omega_0 = (LC)^{-\frac{1}{2}}$, and $C = \epsilon \frac{\Sigma}{d}$. Then $d \sim \sqrt{d}$.

b The major part of the magnetic field flows through $\mathcal{S}$: a period increase changes slightly the resonance frequency.

c $L \sim r_{int}^2$ and $C \sim (r_{ext} - r_{int})^2$, then : $\omega_0 \sim \frac{1}{r_{int}(r_{ext} - r_{int})}$

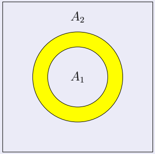


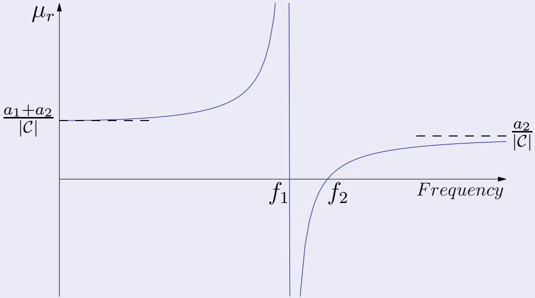
(a) (b) (c)

The polaritonic law

Hypothesis : $h = h_1$ in A_1 and $h = h2\ A_2$, where .

- Fraday's law : $i\omega\Phi + V = 0$, Ampere's law : $\mathcal{I} = h_1 - h_2$.
- Magnetic field's flux : $|\mathcal{C}|B = \mu\left(|A_1|h_1 + |A_2|h_2\right)$ and $\Phi = \mu h_1 |A_1|$
- Capacitor in the airgap : $\mathcal{I} = i\omega CV$.





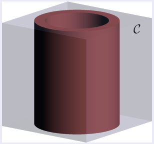
$$\mu_{eff} = \mu\,\frac{a_1 + a_2}{|\mathcal{C}|}\,\frac{\left(\frac{\omega}{\omega_2}\right)^2 - 1}{\left(\frac{\omega}{\omega_1}\right)^2 - 1}$$

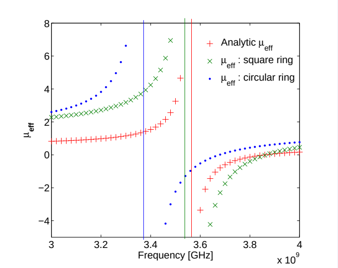
$$\omega_1 = \left(\mu C |A_1|\right)^{-\frac{1}{2}}$$

$$\omega_2 = \sqrt{1 + |A_1|/|A_2|}\;\omega_1$$

Comparison with the simulations

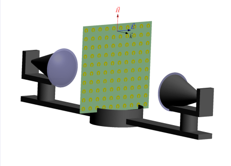
- Simulations of square and circular split-rings, where : $|A_1| = 0.09$ cm^2, $|A_2| = 0.51$ cm^2, $a = 1$ cm^2 and $d = 1$ mm.
- Better agreement of the analytical law with the simulations of the square ring.
- 2D simulations are achieved by having the ring's height equals the cell's height.





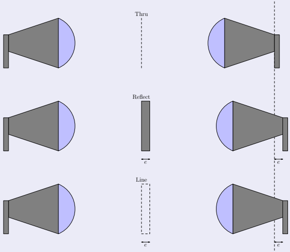
Free space characterization bench

- 2-18 GHz quad ridged horn antennas.
- Rexolite ($\epsilon_r = 2.54$) lenses.
- Gaussian beams are converted to a quasi-planar wave at the middle of the bench.



Thru Reflect Line calibration

Micrometric positioning stages are used to move the second antenna by the thickness of the metal plate (that equals the thickness of the material to be characterized).

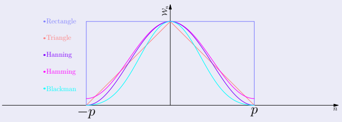


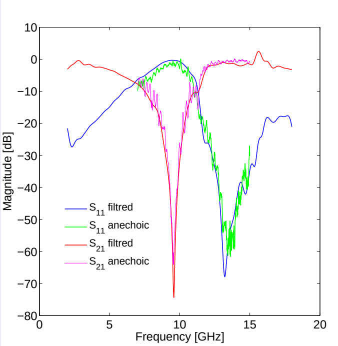
Time domain analysis

- After the calibration, The S parameters are filtered in the time-domain.
- Windows with different shapes could be applied.
- The multiple reflexions on the environment are then eliminated

Rectangular windows in particular induce a signal ringing (Gibbs phenomenon).

Very good agreement with the obtained results in an anechoic chamber (ϵ-negative rods).

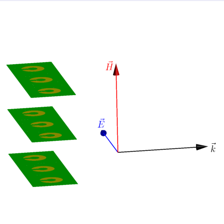


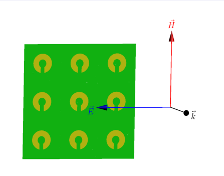


Polarization effect

- The substrate is epoxy : relative permittivity ($\epsilon_r = 4.4$) and height $h = 1.6$ mm.
- Ring's dimensions : $a_{int} = 1.8$ mm, $a_{ext} = 4.4$ mm and $d = 1$ mm.

- Analytical : 11.13 GHz.
- Simulations : 11.09 GHz.
- Perpendicular polarization : 10.9 GHz.
- Parallel polarization : 10.95 GHz

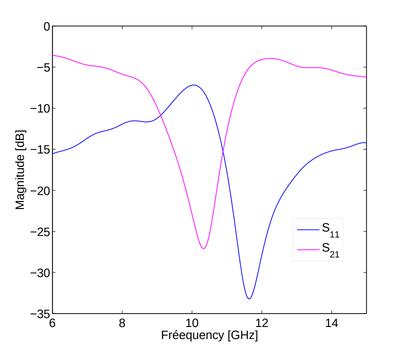


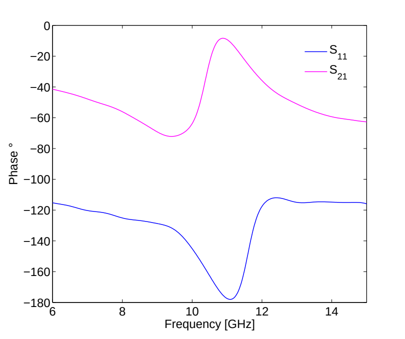


Measurements results

Ring's dimensions : $r_{int} = 1$ mm, $r_{ext} = 2.5$ mm.

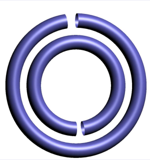
d (mm)	Measured (GHz)	Analytical (GHz)	Simulated (GHz)
0.8	9.4	9.37	9.47
0.9	9.6	9.94	10.05
1	10.32	10.47	10.59

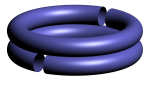




Conclusions

- The effective permeability, with a negative real part, is obtained with a minimal computational cost.
- A simple analytical law model the permeability of 2D SRR.
- Very good agreement with the free space measurements.
- Not adapted to the simulation of double split-ring resonators.





References

1. P. Gay-Balmaz and O. J. F. Martin, *Efficient isotropic magnetic resonators*, Applied Physics Letters, 81(5), 939-941, 2002.
2. A. Bossavit, *Effective frequency-dependent permeability of split-ring metamaterials via homogenization*, IEEE Transactions on Magnetics, 45, 1276-1279, 2009.