Homogenization of Split-Ring Arrays, Seen as the Exploitation of Translational Symmetry

Alain Bossavit[1]

[1] LGEP (CNRS, Univ. Paris Sud), 11 Rue Joliot-Curie, 91192 Gif-sur-Yvette, France

Abstract Homogenization, which reduces the cost of numerical simulations in materials with repetitive structure, is a promising approach to the design of metamaterials. This cost reduction stems from the possibility to compute effective permeability and permittivity of an equivalent homogenized material by solving an auxiliary "cell problem" on the generating cell of the metamaterial. The first part of this paper is a tutorial, where the procedure is described in the context of the exploitation of symmetry via harmonic analysis, and justified by an appropriate asymptotic result when the size of the cell is small enough. The second part argues that this standard approach can fail, and explains how it does, when a second small parameter, besides the cell's size, is present in the physical situation. This is precisely what happens in the case of an array of split rings, where the slit's width competes, so to speak, with the cell's size in the passage to the limit that leads to the cell problem. We show how this competition must be arbitrated in order to recover the negative effective permeability one may expect, on physical grounds, near some resonant frequency, in the case of a split-rings array. A simplified model, amenable to analytical computation, illustrates this "frequency dependent homogenization" procedure.

1 Introduction

When a regular, crystal-like, array of small split rings (Fig. 1, left) is immersed in an AC magnetic field, the metamaterial thus obtained can behave in surprising ways [1]. For instance, the spatial average B of magnetic induction b (at a large enough scale with respect to the size of the "cell" C of Fig. 1), and the spatial average P of the reactive power inside the material relate by $P = i\omega\, (\nu_{eff}\, B)^* \cdot B$ (that is to say, $P = i\omega H^* \cdot B$, if one sets $H = \nu_{eff}\, B$), with an effective reluctivity ν_{eff} that is not only anisotropic (i.e., a tensor), and complex-valued (because of inevitable Joule losses), but can exhibit a *negative* real part in some directions (here, vertically) in some narrow window $[\omega_1, \omega_2]$ of angular frequencies. This is due to an "internal resonance" when $LC \sim \omega^{-2}$, where L is the ring inductance and C the slit capacitance. All it takes is a near-perfect conductor (or dielectric), in order to have $e \sim 0$ in the ring. Then, by Faraday's law, a displacement current

must cross the slit, hence a large electric field there, whose average energy can offset the cell's magnetic energy if ω is slightly above the resonant frequency.

Our objective is to define a so-called "cell problem", a boundary value problem whose equations are those of Maxwell, but set on C only instead of being set on the entire space, which once solved would provide ν_{eff}, or its inverse μ_{eff}, and also (though we shall not go into this in detail) the analogous effective permittivity $\varepsilon_{eff.}$. From this point on, one would be free to solve the macroscopic problem at hand (with a specific shape for the metamaterial-filled region(s), specific source currents, etc.) by using the effective coefficients only, ignoring the microstructure that has already been dealt with when solving the cell problem(s). This will allow a cheaper numerical simulation (by allowing the use of much coarser finite element nets, for instance), and perhaps more importantly, serve in *designing* metamaterials by repetitively solving the cell problem inside some iterative procedure which makes the cell's structure evolve towards some optimal one.

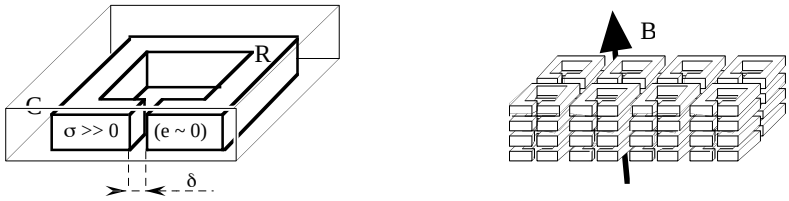

Fig. 1 A "split ring" R in its symmetry cell C (left), and a stack of such cells, making a metamaterial (right). An oversimplistic design, as such materials go, but enough to demonstrate our frequency-dependent homogenization technique. Later, the thin slit of width δ is replaced by a surface Σ, and the air region $C - (R \cup \Sigma)$ is denoted A.

This procedure—solve a micro-scale cell problem, or several, then address a macro-scale problem where all materials are homogeneous, with effective coefficients as found—is called *homogenization* [2]. As an approximation technique (which it is: one replaces the original problem by a different one, deemed close to it), it requires a justification, that only some appropriate convergence theorem can provide. This is our main concern in this work. But let it be clear that *which* theorem to prove (a matter of asking the right question, that is, an act in *modelling*), is more important here than *proving* it, a technical matter.

The first part of the paper is devoted to this proof, which we shall be able to keep simpler than in other approaches to homogenization (such as matched asymptotic expansions [2], or Gamma-convergence [3]) by assuming *linearity* of the constitutive laws. This allows one to use standard harmonic analysis mechanisms such as (spatial) Fourier transform and Floquet–Bloch decomposition [4], whose relatively involved interrelations provide the gist of the proof technique. Shortly said, one designs a family of *virtual* problems P^{α}, in which the *actual*, physical one is embedded (we make it correspond to $\alpha = 1$), and one

proves convergence, in some weak sense, of the solution u^α of P^α to the solution u^0 of some limit problem P^0, an inspection of which exposes the cell problem. When α models the cell size, P^0 can be shown to correspond to a homogeneous medium, which can then be substituted for the periodic array if u^1 (the actual solution) and u^0 are close enough, which one may usually assume for small-grained arrays.

But this very assumption must be questioned in the case of metamaterials, as we shall see. It remains possible to embed the real problem P^1 in a family $\underline{P}^\alpha$ of virtual ones, and to ensure the closeness between u^1 and $\underline{u}^0$, but for this one must account for the relation that must exist, at the intended working frequency, between cell's size and slit's width for the internal resonance to occur. Hence the introduction in the asymptotic analysis of a second small parameter, that must be linked with the cell's size in a way we shall discover, thus making the $\underline{P}^\alpha$ family—and the cell problem derived from its limit $\underline{P}^0$ — different from the P^α and P^0, and cell problem, that fit a low-frequency situation. A convergence result still holds, and can be proven by similar means, but the final result—effective coefficients, and how they depend on ω —is radically different. These modelling issues make the second part of the paper.

The organization is as follows: Section 2, recalling elements of harmonic analysis on groups [5], shows how the use of both spatial Fourier transform and Bloch–Floquet transform pertains to the exploitation of symmetry (translational symmetry, complete or partial) and explains their shared features by this commonality. Section 3 addresses their differences and how they relate nonetheless, suggesting that Fourier analysis is a limit case of Floquet–Bloch, as obtained when the size of the symmetry cell tends to 0. Section 4 uses this to establish the main convergence result about homogenization in magnetostatics. In Section 5, the same kind of result is obtained for the full Maxwell system (in steady-state regime at angular frequency ω), but is found, though correct, inadequate as regards modelling. Section 6 solves the conundrum this raises in the case of the split-ring metamaterial by introducing a second parameter, besides the cell's size, in the analysis, namely the slit's width. A summing-up discussion of such "two small-parameter" problems makes Section 7.

About notation: We shall distinguish *vectors*, elements of the 3D vector space V_3, and *points*, elements of the associated affine space, denoted A_3. If x is a point and v a vector, $x + v$ denotes a point, the *translate* of x by v. This way, V_3 is an additive group, which thus *acts* on A_3 "by translations". (Italics, when not used for emphasis, signal that some implicit definition is being given, or suggested, as it is the case here.) $L^2(M)$ denotes the Hilbert space of square-integrable functions over the measure space M. Complex conjugation is indicated by a star.

2 Harmonic Analysis and the Exploitation of Symmetry

Homogenization is the exploitation of symmetry, namely translational symmetry of the crystal-like structure, taking account of the smallness of the symmetry cell. These are two distinct issues. The present Section addresses the first one of these, leaving smallness of the cell to be considered later in Section 3.

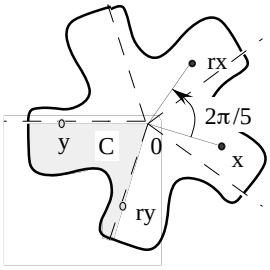

Fig. 2 Domain D with fivefold symmetry, generated by rotations r^k of the symmetry cell C. Note how r relates pairs of "new boundary" points, like here y and ry.

Let us begin with a case of rotational symmetry (Fig. 2), where the number of symmetry operations is finite, before moving to translational symmetry, where the group of symmetry operations is infinite (which will raise technical, if not essential, difficulties). In domain D with n–fold rotational symmetry (n = 5 here), one wishes to solve the problem $-\mathrm{div}(\mu \,\mathrm{grad}\, \varphi) = q$, for a given magnetic charge q, with $\varphi = 0$ on ∂D. We shall call this "problem P". (Charge q is fictitious, physically. The source of the field is rather a current density, or a magnetization. But there is always an equivalent charge, then, which is the one we deal with.) Let r denote the $2\pi/n$ rotation around the axis, rx the rotated image of a point x, note that $rD = D$, and that there is a "symmetry cell" C (which can be carved out in many different ways) such that D be the union, with minimum overlap, of its transforms r^kC, k = 0 to 4. Assume that $\mu(rx) = \mu(x)$ for all x (apart, possibly, from those of some zero-measure set: we shall gloss over this, which raises no difficulty). These identities also hold for all elements $g = r^k$, k = 0 to 4, of the group G generated by r. We shall use these peculiarities to replace problem P by five problems P_k, independent, all set on C instead of D (which makes each P_k "simpler" than P, in an obvious way), each with its own set of boundary conditions on the "new boundary" $\partial C - \partial D$. The trade-off (five "small" problems instead of a single "big" one), is obviously favorable in the case of, e.g., finite element computations. The method requires the following machinery. (See [6] for further details, and for how this transposes to discrete formulations, using finite elements.)

One calls *characters* of G the complex-valued functions χ such that $\chi(gh) = \chi(g)\chi(h)$ for all g, h in G and $\chi(\mathrm{id}) = 1$ for the identity transform. Since r^5 is the identity, this implies, as easily verified, that there are five characters χ_k, for

integers k from 0 to 4, fully determined by $\chi_k(r) = \exp(2i\pi k/5)$. Note that $\chi_j\chi_k = \chi_{j+k}$, if one counts $j + k$ modulo 5, so that characters form themselves a group Γ, called the *dual* of G. (The dual of Γ is of course G.) Given a function φ, denote by $U_g\varphi$ the function defined by $(U_g\varphi)(gx) = \varphi(x)$. Then form, for each k, the *projector* $\Pi_k = n^{-1}\sum_{g\in G}\chi_k^*(g)\,U_g$, thus called because $\Pi_k\Pi_k = \Pi_k$ and $\Pi_j\Pi_k = 0$ if $j \neq k$. We note that $\varphi = \sum_k \Pi_k\varphi$, and that $\mathrm{div}(\mu\,\nabla\Pi_k\varphi) = \Pi_k[\mathrm{div}(\mu\,\nabla\varphi)]$, the property known as G-*equivariance* of the differential operator $\mathrm{div}(\mu\,\mathrm{grad})$. All this shows that "problem Q_k", that is, solving $-\mathrm{div}(\mu\,\nabla\varphi) = \Pi_k q$ over all D with $\varphi = 0$ on ∂D, has $\Pi_k\varphi$ for its solution. Moreover, $\Pi_k\varphi$ transforms in a particular way under rotation, owing to the obvious equality $U_r\Pi_k = \chi_k(r)\Pi_k$: one has $(\Pi_k\varphi)(rx) = \chi_k(r)(\Pi_k\varphi)(x)$, which shows that $\Pi_k\varphi$ is determined by its restriction φ_k to the symmetry cell, and that φ_k satisfies "problem P_k": $-\mathrm{div}(\mu\,\nabla\varphi) = \Pi_k q$ over C, with $\varphi = 0$ on ∂D and $\varphi(rx) = \chi_k(r)\varphi(x)$ for pairs of points $\{x, rx\}$ that belong to the new boundary.

Hence the recipe: (1) Form the right-hand sides $\Pi_k q$ by computing $\Pi_k q(x) = n^{-1}\sum_j \exp(-2i\pi kj/n)\,q(r^j x)$ for points x of C, the step called "(discrete) Fourier analysis", (2) Solve each problem P_k over C, using on the new boundary the "pseudo-periodic boundary condition" (p.p.b.c., from now on) $\varphi(rx) = \exp(2i\pi k/n)\varphi(x)$, (3) Extend the solution φ_k of P_k to all D, using $\varphi_k(rx) = \exp(2i\pi k/n)\,\varphi_k(x)$, hence $\Pi_k\varphi$, and sum up to recover $\varphi = \sum_k \Pi_k\varphi$. (In particular, $\varphi = \sum_k \varphi_k$ over C.) This step is "(discrete) Fourier synthesis".

Let us now test the power of this method by considering the case of crystal-like symmetry. This time, $-\mathrm{div}(\mu\,\nabla\varphi) = q$ is to be solved over all space (a well-posed problem if $q \in L^2(A_3)$ has bounded support), and $\mu(x + \tau) = \mu(x)$ for all points x and all translation vectors τ of the form $\sum z^i v_i$, where v_1, v_2, v_3 are three fixed independent vectors and the z^i are relative integers. The set T of such translation vectors, which do form a group of mappings from A_3 to itself (a subgroup of V_3, isomorphic to Z^3), is called the *Bravais lattice.* There is no unique, or even preferred choice for the cell C: We select an arbitrary spatial point c (for *center*) and take for C the set of points $x = c + \sum_i \lambda^i v_i$ for all λ^i such that $|\lambda^i| \leq 1/2$. The translates of C by all vectors τ of T pave space, with overlap at their boundaries. Characters are here functions $\chi_\kappa(\tau) = \exp(i\,\kappa\cdot\tau)$, where κ is a 3D vector, but not all such vectors are needed: Let's call w_1, w_2, w_3 the *dual* vectors to the v_is, defined by $w_i \cdot v_i = 2\pi$ and $w_j \cdot v_i = 0$ if $i \neq j$, and B (akin to the "Brillouin zone" of crystallography) the set of vectors $\sum_i \lambda^i w_i$ with $|\lambda^i| \leq 1/2$. Translates of B by vectors $\theta = \sum_i z^i w_i$, with integer weights z^i, pave V_3. Since $\exp(i\kappa\cdot\tau) = \exp(i\,(\kappa + \theta)\cdot\tau)$ for such vectors θ, one gets all the characters by letting κ span B only. Moreover, pairs of points of ∂B that are related by such θ-translations yield the same character. Therefore, the dual group of T is B "wrapped around", with identification of such points, a 3D *torus* (which we still call B). This time, group T and its dual are very different: Both have an infinity of elements, but T is discrete, topologically speaking, whereas B is continuous and compact. (Predictably, the dual of B is T: The corresponding Fourier technique, which shall not be detailed here, is that of *Fourier series,* as

applied to functions living on the torus T, that is to say, to triply periodic functions of three variables.)

The order of T being infinite mandates minor adjustments with respect to the above example (no division by n). For x in C, set

$$(\Pi_\kappa\varphi)(x) = \mathrm{vol}(C) \sum_{\tau\in T} \exp(-i\kappa\cdot\tau)\, \varphi(x+\tau).$$

Since $(\Pi_\kappa\varphi)(x+\tau) = \exp(i\kappa\cdot\tau)\,(\Pi_\kappa\varphi)(x)$, as easily checked, this is a pseudo-C-periodic function. Then $\varphi = (2\pi)^{-3}\int_B d\kappa\, \Pi_\kappa\varphi$, which suggests the same kind of approach as in the case of cyclic symmetry: Since equivariance obviously holds, form the functions $\Pi_\kappa q$ on C, solve

$$-\mathrm{div}(\mu\nabla\varphi) = \Pi_\kappa q, \qquad \text{with p.p.b.c.,} \tag{1}$$

for all κ, hence φ_κ on C, then integrate in κ to recover φ. This is *Floquet–Bloch* analysis.

But now prospects to save on computation look grim: What is the point of replacing a single partial differential equation (even though its domain is the whole space), by an infinity of boundary value problems on C which—depending on κ by their boundary conditions, as they do—are all different? We note that the treatment by the Fourier transform, when applicable, that is, when μ is invariant by all translations, does not suffer from the same shortcomings, because it replaces the original equation by an infinity of *algebraic* ones (namely, $\kappa\cdot(\mu\,\hat{\varphi}(\kappa)\,\kappa) = \hat{j}(\kappa)$, for all $\kappa\in V_3$, where the Fourier component $\hat{j}(\kappa)$ and $\hat{\varphi}(\kappa)$ of j and φ are complex *numbers*, rather than *functions*). These equations can be solved in one stroke, hence the solution φ by inverse Fourier transform. What homogenization does, as we shall see, is reduce the complexity of the Floquet–Bloch treatment by solving (approximately, of course) all problems (1) in one stroke.

3 Bloch and Fourier

To make easier the comparison between Bloch and Fourier analyses, and also to make formulas look more familiar, we take point c as origin of a reference frame based on the v_is, and reuse symbol x to denote the translation vector $x - c$. Set

$$\hat{\varphi}_\kappa(x) = \exp(-i\,\kappa\cdot x)\,(\Pi_\kappa\varphi)(x).$$

This way, the "Bloch mode" $\hat{\varphi}_\kappa$ is C-periodic and is obtained by the summation

$$\hat{\varphi}_\kappa(x) = \mathrm{vol}(C) \sum_{\tau\in T} \exp(-i\kappa\cdot(x+\tau))\, \varphi(x+\tau). \tag{2}$$

The inverse transform formula becomes, accordingly,

$$\varphi(x) = (2\pi)^{-3} \int_B d\kappa \exp(i\,\kappa \cdot x)\, \hat{\varphi}_\kappa(x). \tag{3}$$

These formulas compare with the direct Fourier transform,

$$\hat{\varphi}(\kappa) = \int_{V_3} d\tau \exp(-i\,\kappa \cdot \tau)\, \varphi(\tau), \tag{4}$$

and the inverse one:

$$\varphi(x) = (2\pi)^{-3} \int_{V_3} d\kappa \exp(i\,\kappa \cdot x)\, \hat{\varphi}(\kappa). \tag{5}$$

The right-hand side (r.h.s.) of (2) looks like an approximate quadrature formula for the r.h.s. of (4), and (5) looks like what (3) would be, should the cell C reduce to a point and B grow up to fill space entirely.

To make rigorous sense of these observations, let us consider a family of Bravais lattices T_α, indexed by a real $\alpha > 0$, generated by the translation vectors αv_1, αv_2, αv_3, and look at what happens when $\alpha \to 0$. The homothetic image C_α of C, with center c and ratio α, makes a symmetry cell for T_α, and the corresponding set of characters is the torus formed from $B_\alpha = \alpha^{-1} B$. We denote by $\langle \hat{\varphi}_\kappa \rangle_\alpha$ the average of $\hat{\varphi}_\kappa$ over C_α. One easily proves that

Lemma 1. *Given* φ *in* $L^2(A_3)$, $\langle \hat{\varphi}_\kappa \rangle_\alpha \to \hat{\varphi}(\kappa)$ *as* $\alpha \to 0$.

In words: For a *fixed* value of the "wavevector" κ, the cell-average of the Bloch mode of a well-behaved function φ converges to its Fourier coefficient $\hat{\varphi}(\kappa)$. There is a kind of reciprocal result, the technical proof of which we omit:

Lemma 2: *Given* φ *and a bounded family* $\{\varphi^\alpha : \alpha > 0\}$ *of functions in* $L^2(A_3)$, *assume that* $\langle \hat{\varphi}^\alpha{}_\kappa \rangle_\alpha \to \hat{\varphi}(\kappa)$ *for all* κ *when* $\alpha \to 0$. *Then* φ^α weakly *converges to* φ *when* $\alpha \to 0$.

In words: If cell averages of Bloch modes of φ^α converge towards Fourier coefficients of φ, for all values of κ, then $\int \varphi^\alpha \psi \to \int \varphi \psi$ for all ψ (this is what weak convergence means), so weighted averages of φ^α converge towards those of φ. Remark than no more is expected from homogenization than good approximation of spatial *averages* of fields, so this looks fine.

4 Static Homogenization: Magnetostatics

Rather than committing ourselves to the use of a scalar magnetic potential φ as we did thus far, it will be more convenient to deal with the problem in the

following symmetric form: Given μ, invariant by all translations of T, and given the compactly supported source current j, find fields h and b such that

$$\operatorname{div} b = 0, \qquad b = \mu h, \qquad \operatorname{rot} h = j. \tag{6}$$

This is "problem P". We shall search for the Floquet–Bloch decompositions of b and h, that is, for functions $\hat{h}_\kappa$, $\hat{b}_\kappa$, living on the symmetry cell C, such that $h(x) = (2\pi)^{-3}\int_{V_3} d\kappa \exp(i\,\kappa\cdot x)\,\hat{h}_\kappa(x)$, etc. Since $\operatorname{rot}[\exp(i\kappa\cdot x)\,\hat{h}_\kappa(x)]$ is the same as $\exp(i\kappa\cdot x)\,[\operatorname{rot} + i\,\kappa\times]\,\hat{h}_\kappa(x)$, with obvious notation, and since a similar relation holds about div, one has "problem P_κ" to solve,

$$(\operatorname{div} + i\,\kappa\cdot)\,\hat{b}_\kappa = 0, \qquad \hat{b}_\kappa = \mu\,\hat{h}_\kappa, \qquad (\operatorname{rot} + i\,\kappa\times)\,\hat{h}_\kappa = \hat{j}_\kappa, \tag{7}$$

for each $\kappa \in B$, on C, with periodic boundary conditions, where the $\hat{j}_\kappa$'s are the Bloch modes of j. Notice how the multiplication by $\exp(-i\,\kappa\cdot x)$, while uniformizing the boundary conditions for this κ-indexed infinity of cell problems, moved the dependence on κ into the differential operators. Thus (7) is by no means simpler than (6), quite the contrary.

But now comes the decisive move, by which we exploit the smallness of the cell C. Let us embed problem P into a family P^α of similar ones, but with C_α-periodicity of μ instead of C-periodicity, so that (6) is problem P^1, one among the family. The corresponding cell problems $P^\alpha{}_\kappa$ are the same as in (7), apart from the α indexes, and the C_α- instead of C-periodicity. As this last feature prohibits a comparison between the $P^\alpha{}_\kappa$'s for a fixed κ and different α's, we "pull back" problems $P^\alpha{}_\kappa$ onto C by the transform $x \to c + (x - c)/\alpha$, which blows up C_α to C. Denoting by $b^\alpha{}_\kappa$, $h^\alpha{}_\kappa$, etc., hats dropped, the pullbacks (e.g.: $b^\alpha{}_\kappa(y) = \hat{b}^\alpha{}_\kappa(c + \alpha(y - c))$), thus living on C, and C-periodic), one sees that

$$(\operatorname{div} + i\,\alpha\kappa\cdot)\,b^\alpha{}_\kappa = 0, \quad b^\alpha{}_\kappa = \mu\,h^\alpha{}_\kappa, \quad (\operatorname{rot} + i\,\alpha\kappa\times)\,h^\alpha{}_\kappa = \alpha\,j^\alpha{}_\kappa, \tag{8}$$

with now the same μ for all (it's the original μ, C-periodic), and the prospect to see all these κ-subproblems become "the same" when $\alpha \to 0$.

Proposition 1. *When* $\alpha \to 0$*, the solution* $\{b^\alpha{}_\kappa, h^\alpha{}_\kappa\}$ *of* (8) *converges* (in the strong sense of $L^2(C)$) *towards the solution* $\{b_\kappa, h_\kappa\}$ *of*

$$\operatorname{div} b = 0, \qquad b = \mu h, \qquad \operatorname{rot} h = 0 \tag{9}$$

$$i\,\kappa\cdot\langle b\rangle = 0, \qquad i\,\kappa\times\langle h\rangle = \hat{j}(\kappa). \tag{10}$$

where $\langle\ \rangle$ *denotes averaging over* C.

Proof (sketched). Terms in α vanish, hence (9). Since $b^\alpha{}_\kappa$ and $h^\alpha{}_\kappa$ are C-periodic, the integrals $\int_C \operatorname{div} b^\alpha{}_\kappa$ and $\int_C \operatorname{rot} h^\alpha{}_\kappa$ are null. Integrate the first line of (8) over C, and use Lemma 1 to find the right-hand side $\hat{j}(\kappa)$ in (10). ◊

Now look at (9)(10). This problem still depends on κ, but it splits into two parts. One is "find the relation $B = \mu_{eff} H$ that must exist between vectors B and H for

$$\text{div } b = 0, \quad \langle b \rangle = B, \quad b = \mu h, \quad \text{rot } h = 0, \quad \langle h \rangle = H \tag{11}$$

to have a solution", and this is the expected *cell-problem*, from which κ has disappeared. The second part consists in finding, for each κ, vectors $\langle b \rangle$ and $\langle h \rangle$ (complex valued) such that $i\kappa \cdot \langle b \rangle = 0$, $\langle b \rangle = \mu_{eff} \langle h \rangle$, $i\kappa \times \langle h \rangle = \hat{j}(\kappa)$, and here we recognize the κ-indexed algebraic problems associated with the Fourier method of solving

$$\text{div } b = 0, \qquad b = \mu_{eff} h, \qquad \text{rot } h = j. \tag{12}$$

Thanks to Lemma 2 (and the fact that the averages of a field over C_α and of its pullback over C are equal), we may conclude as follows:

Theorem 1. *When* $\alpha \to 0$, *the solution* $\{b^\alpha, h^\alpha\}$ *of problem* P^α *weakly converges towards the solution* $\{b, h\}$ *of* (12), *where the homogenized permeability* μ_{eff} *is the one obtained by solving the cell problem* (11).

Such an asymptotic result was a precondition to replacing problem P^1, the original one, by P^0, if we so label problem (12), so this is satisfying. But does it *justify* this replacement? As with all perturbative techniques, this depends on the magnitude of the terms thus neglected in the Taylor expansion in α of $\{b^\alpha, h^\alpha\}$ near $\alpha = 0$. It can be shown (but this is another issue than what Thm 1 addresses) that this is so if the spectral content of the spatial Fourier transform of j is poor in "small spatial wavelengths", where small refers to the size L of cell C, i.e., if $\hat{j}(\kappa)$ is small enough to be neglected when $L|\kappa| << 1$ does not hold. To appreciate whether this is so belongs to the "modelling" part of the procedure.

5 Homogenizing the Full Maxwell System

Let us now try the same approach on the Maxwell equations, $-i\omega\varepsilon e + \text{rot } h = j$ and $i\omega\mu h + \text{rot } e = 0$, where ε integrates the conductivity σ by the usual trick of setting $\varepsilon = \varepsilon_0 - i\sigma/\omega$. Our work is cut out for us: Proceeding exactly as above, we find this instead of (8):

$$-i\omega\alpha d^\alpha_\kappa + (\text{rot} + i\,\alpha\kappa\times) h^\alpha_\kappa = \alpha j^\alpha_\kappa, \qquad d^\alpha_\kappa = \varepsilon\, e^\alpha_\kappa,$$

$$i\omega\alpha b^\alpha_\kappa + (\text{rot} + i\,\alpha\kappa\times) e^\alpha_\kappa = 0, \qquad b^\alpha_\kappa = \mu\, h^\alpha_\kappa,$$

and the $\alpha = 0$ limit is characterized by

$$\begin{aligned} &\operatorname{div} b = 0, && b = \mu h, && \operatorname{rot} h = 0, \\ &\operatorname{div} d = 0, && d = \varepsilon\varepsilon, && \operatorname{rot} e = 0, \\ &-i\omega \langle d\rangle + i\,\kappa \times \langle h\rangle = \hat{j}(\kappa), && i\omega \langle b\rangle + i\,\kappa \times \langle e\rangle = 0. && (13) \end{aligned}$$

The weak limit, the same theorem tells us, will satisfy (13) with constitutive relations $\langle b\rangle = \mu_{\text{eff}} \langle h\rangle$ and $\langle d\rangle = \varepsilon_{\text{eff}} \langle e\rangle$, which are the image in Fourier space of the Maxwell equations for a material with effective coefficients μ_{eff} and ε_{eff}, as given by solving the cell problem (11) and its electrostatic counterpart.

But this is a big disappointment! We *don't* expect the effective coefficients for a metamaterial to be independent of frequency, as this result would tend to suggest, especially not in the case of Fig. 1. What went wrong? Not the logic underlying the theorem, but the modelling: The choice of problems P^α in which to embed P^1 cannot be the same as in statics, because it must take into account the existence, besides the cell's size, of a second small parameter, the width of the slit (to say nothing of the penetration depth, which we take null from the onset by assuming $e = 0$ in the bulk of the ring).

6 Homogenizing the Split-Ring Metamaterial

Indeed, since the resonance condition $LC \sim \omega^{-2}$ is essential, it should be an invariant feature of all problems P^α in order to be preserved at the limit. When $\alpha \to 0$, both L and C scale like α if the cell is shrunk by homothetic contraction, hence the resonance is lost, so one should imagine a family of problems P^α that make the slit capacitance behave in $1/\alpha$. This is easy: Take the slit width in P^α equal to $\alpha^3\delta$. Let us now look for the $\alpha = 0$ limit under these conditions.

To keep things simple, we start from a modelling of problem P^1 that already acknowledges the high conductivity of the ring (this is done by assuming $e = 0$ in R) and the smallness of δ (this is done by modelling the slit by a surface Σ that bears a capacitive layer). Cell C is made of the ring R, the slit Σ, and the "air part" A around. With hopefully little risk of confusion, we also denote by R, Σ, A the set-unions of all translates of these cell parts in the given lattice. The same conventions apply, with index α appended, for problems P^α. Now, the version of problem P^α we propose ourselves is, in weak form, *find* h^α *such that*

$$\begin{aligned} &\int_{A_\alpha} i\omega\mu\, h^\alpha \cdot h' + \int_{A_\alpha} (i\omega\varepsilon)^{-1} (\operatorname{rot} h^\alpha - j) \cdot \operatorname{rot} h' \qquad (14) \\ &\quad \ldots + \int_{\Sigma_\alpha} (i\omega\varepsilon)^{-1} \alpha^3 \delta\, (n \cdot \operatorname{rot} h^\alpha)\, (n \cdot \operatorname{rot} h') = 0 \end{aligned}$$

for all test fields h', where n denotes the unit normal on Σ_α. Notice how (in the case $\alpha = 1$) the third term accounts for the capacitive effect across Σ.

We proceed as above, by throwing the Bloch decomposition of h into (14), hence the κ-indexed cell problems, then scaling to pull them back to the reference cell C. This results in *find* h^α_κ (a Bloch mode, now), C-*periodic, such that*

$$\alpha^3 \int_A i\omega\mu\, h^\alpha \cdot h' + \alpha \int_A (i\omega\varepsilon)^{-1} ((\mathrm{rot} + i\alpha\kappa \times) h^\alpha - \alpha j) \cdot (\mathrm{rot} + i\alpha\kappa \times) h'$$

$$\ldots + \alpha^3 \int_\Sigma (i\omega\varepsilon)^{-1} \delta\, [n \cdot (\mathrm{rot} + i\alpha\kappa \times) h^\alpha][n \cdot (\mathrm{rot} + i\alpha\kappa \times) h'] = 0$$

for all test fields h', where one clearly sees how the α^3 term in front of δ maintains the balance between inductive and capacitive reactive powers. The second term acts like a penalty one, making rot h equal to 0 in the $\alpha = 0$ limit.

Let us now, leaving intermediate details aside, describe this limit. Since rot $h = 0$ in A, we write it $h = \mathrm{grad}\, \varphi$, with φ possibly multivalued, since a current I will flow in the ring across Σ. Let us therefore introduce a "cut" S (cf. Fig. 3) across which φ may have a jump $[\varphi]$, equal to I. Jumps can also exist on ∂C, because although h is C-periodic, φ need not be: What one must have is just $\varphi(x + v_i) - \varphi(x)$ equal to some constant c_i for pairs of boundary points x and $x + v_i$. Call Φ the space of admissible potentials under these conditions. The weak form of the cell problem then turns out to be, *find* φ *in* Φ *such that, for all test functions* φ' *in* Φ,

$$\int_A i\omega\mu\, \nabla\varphi \cdot \nabla\varphi' + (i\omega C)^{-1} [\varphi][\varphi'] = \int_A i\omega\, B \cdot \nabla\varphi', \qquad (15)$$

where C, the capacitance of the slit, is taken equal to $\int_\Sigma \varepsilon_0/\delta$, and B, a given vector parameter, can be recognized as the average induction. To find the effective reluctivity, as suggested in the Introduction, we then set $\nu_{\mathrm{eff}}\, B \cdot B = \int_A \mu |\nabla\varphi|^2 - 1/(C\omega^2) |[\varphi]|^2$. Note how ν_{eff} can change sign. It is *real*, here, owing to our perfect conductor assumption, which excludes losses, but it would be easy to correct (15) to account for these: Just add a "boundary impedance" term, in $(1 + i)(\sigma d)^{-1}\, h \cdot h'$, where d is the skin depth, on ∂R.

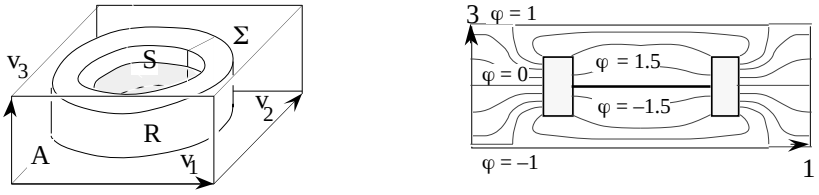

Fig. 3 Left: Surface Σ modelling the slit, and cutting surface S. Right: Typical behavior of the solution φ of the cell problem in an axial plane (arbitrary figures). Notice the jump of φ across the cut and the opposite directions of the magnetic field inside and outside the ring.

This is a mildly exotic variant of the magnetostatics problem, easy to solve by using nodal elements for φ in A (and node-doubling on the cut S). If one focuses interest on the vertical part of ν_{eff} (B then being vertical), and if the

height of the ring almost matches that of the cell, the two-dimensional situation that results is amenable to analytical calculation (Fig. 4). One finds $\nu_{eff}(\omega) = \nu_0(1 - \omega^2 \mu_0 aC)/[a + a'(1 - \omega^2 \mu_0 aC)]$, where a and a' are the *relative* (to the cross section) areas of the two parts of the air region and C the slit's capacitance. Hence a window $\omega_1 = \nu_0/aC$ to $\omega_2 = \nu_0(a + a')/aa'C$ of negative effective reluctivity. For σ finite but very large, correcting for skin effect as explained above, one can expect the result plotted on Fig. 4, right

Fig. 4 Left: Horizontal cross section of C. The ring current I is H – H' (by Ampère), one has I = iωCV, where V is the voltage drop across the gap, and V relates to H by Faraday. Hence the formula for $\nu_{eff}(\omega)$ given below. Right: Plot of ν_{eff} in the complex plane, parameterized by the angular frequency, if Joule losses are accounted for.

7 Discussion

Attentive readers may well have suspected a sleight-of-hand in what precedes: I first embedded the physical problem P in a family P_α, indexed by the cell-size-related parameter α, searched for the limit P_0, and then... threw away the result as non-satisfactory, just to select, seemingly out of the blue, *another* family of problems, which I decided was more suitable!! What's going on?

This is a delicate issue, but one that is not special to homogenization techniques (see e.g. [7], or the book [8]), and it can be discussed in non-specific terms. Suppose we have at hand a real-life problem where two small parameters a and b make numerical simulations difficult, but are also, both of them, good candidates to perturbative treatment. To formalize such an approach, let us consider the family of problems $P_{\alpha\beta}$ where a and b are replaced, all other things equal, by αa and βb, and where $q_{\alpha\beta}$ denotes the value of some quantity of interest, a function of the solution $u_{\alpha\beta}$. The latter may behave wildly when both α and β go to zero, but *if* we can prove, by mathematical arguments, that $q_{\alpha\beta}$ has a limit q_{00}, which can be computed from the solution u_{00} of some limit problem P_{00}, and that q is smooth in $\{\alpha, \beta\}$ near $\{0, 0\}$, then solving P_{00} instead of the real-life problem P_{11} can be a smart move. We just need to make sure that the derivatives $\partial_\alpha q$ and $\partial_\beta q$ at $\alpha = 0$ and $\beta = 0$ are small enough to

justify the neglect of all terms except the very first one in the Taylor expansion $q_{\alpha\beta} = q_{00} + \alpha\,\partial_\alpha q + \beta\,\partial_\beta q + \ldots$ when $\alpha = 1$ and $\beta = 1$.

In the case of the split-ring problem, a is the cell's size and b the slit's width, and q can be, say, the average magnetic energy density (or rather, since it's complex-valued, its imaginary part, which yields the real part of μ_{eff}). But it's clear, on physical grounds, that q is *not* smooth, not even continuous, at point $\{0, 0\}$, which forbids the simple-minded perturbative approach in which $\{\alpha, \beta\}$ would be treated as a single, if two-dimensional, parameter.

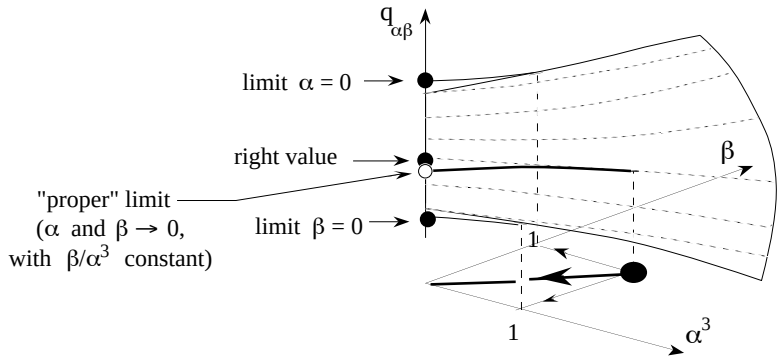

Fig. 5 Graph of the observable parameter q. Though q smoothly depends on α and β, the limits q_{10} and q_{01} can be very different, because of the discontinuity at the origin in parameter space.

On the other hand, perturbation with respect to α alone, or to β alone, is possible. The latter consists in replacing the slit by a surface Σ bearing the capacitance $C = \int_\Sigma \varepsilon_0/\delta$, and letting α stay equal to 1, which ignores the repetitive structure of the metamaterial: No homogenization, therefore. The other option, keeping $\beta = 1$ and letting α go to 0, which ignores the resonance condition $LC \sim \omega^{-2}$, corresponds to the "static" homogenization performed in Section V, and found wanting. Fig. 5 is meant to suggest why these perturbative approaches, *"justified" by a convergence theorem as they may be,* will fail in practice: The derivatives $\partial_{\alpha\beta} q$ at $\{1, 0\}$ and $\{0, 1\}$ are just too large. The correct perturbative approach is suggested by the figure: Find a path $\beta = \psi(\alpha)$, in the $\{\alpha, \beta\}$-plane, that follows, as closely as possible, the projection of the level line passing through $\{1, 1, q_{11}\}$, find the $\alpha = 0$ limit of this family of problems, solve that. Of course we have no way to predetermine this optimal path, but the one we did choose, $\beta = \alpha^3$, is our best guess, because it preserves the feature of problem P_{11} that we know is responsible for the global behavior of the metamaterial, namely the internal resonance within each cell that occurs when $LC \sim \omega^{-2}$. Again, a detailed computation of the derivative $\partial_{\alpha\beta} q$ at the origin, along the curve $\beta = \alpha^3$, would give a posteriori justification, should we need it.

So the homogenization of metamaterials at high frequencies, as a numerical technique, rests on two pillars: Correct modelling, which suggests *which*

convergence result is required to justify the derivation of the cell problem to be solved, and mathematical analysis to establish this convergence. We have seen how misleading the neglect of the former in favor of the latter could be, so it's only fair to insist on the necessity of a convergence theorem: This is, to say things rapidly, what guarantees the smoothness, *outside the singular point at the origin*, of the graph of q, on Fig. 5. Owing to the fact that only *weak* convergence was proven, not all observable quantities will behave that smoothly. For instance—a trivial counter-example—the value of the electric field at point c, or any other fixed spatial point, stands no chance to converge when $\{\alpha, \beta\}$ tends to $\{0, 0\}$, whichever way. Only those observables that are spatial averages of some local function of the actual, highly oscillatory field, will have such a smooth graph. Only parameters of this kind, which fortunately includes the effective coefficients we are after, can legitimately be approximated by homogenization techniques.

References

[1] S. Zouhdi, A. Sihvola, M. Arsalane (eds): **Advances in Electromagnetics of Complex Media and Metamaterials** , NATO Science Series II, 89, Kluwer (Dordrecht), 2002.

[2] A. Bensoussan, J.L. Lions, G. Papanicolaou: **Asymptotic methods for periodic structures**, North Holland (Amsterdam), 1978.

[3] A. Braides: "Introduction to Homogeneization and Gamma-convergence", in A. Braides and A. Defranceschi (eds.): **Homogenization of Multiple Integrals,** Oxford U.P. (Oxford), 1998.

[4] D. Sjöberg, C. Engström, G. Kristensson, D.J.N. Wall, N. Wellander: "A Floquet–Bloch Decomposition Of Maxwell's Equations Applied To Homogenization", **Multiscale Model. Simul., 4,** 1 (2005), pp. 149-71.

[5] G.W. Mackey: "Harmonic Analysis as the Exploitation of Symmetry—A Historical Survey", **Bull. AMS, 3**, 1, Pt. 1 (1980), pp. 543-698.

[6] A. Bossavit: "Boundary value problems with symmetry, and their approximation by finite elements", **SIAM J. Appl. Math., 53,** 5 (1993), pp. 1352-80.

[7] S. Richardson: "How Not to Tackle Some Singular Perturbation Problems", **SIAM Rev., 32,** 3 (1990), pp. 471-73.

[8] W. Eckhaus: **Asymptotic Analysis of Singular Perturbations,** North-Holland (Amsterdam), 1979.