

Frequency-Dependent Homogenization: Application to Metamaterials

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Abstract.

A new approach to homogenization of spatially periodic composite materials is proposed, in which the *frequency* of the excitation is an essential parameter. An analysis of the "cell problem", by which effective homogeneous parameters ϵ_{eff} and μ_{eff} are obtained, show they can have *negative* real part, which is one of the interesting emergent properties of some (so-called "left-handed") metamaterials. The method can be used to *design* such materials, by numerical shape-optimization.

1. Metamaterials challenge standard homogenization theory

Homogenization of constitutive laws is a well established technique [1] when dealing with composite materials with spatial periodicity of constitutive coefficients (here, ϵ , σ et μ). It consists in computing, from the detailed description of the periodicity "cell" that generates the material, by translations, the constitutive parameters of an "equivalent material", *homogeneous* (but not isotropic, as a rule). These parameters, denoted ϵ_{eff} , μ_{eff} , etc., in what follows, can be used for a large-scale computation, ignoring the fine structure, to which one can return locally, at points of special interest, by a corrective calculation.

Homogenization provides an artificial homogeneous material that, were it substituted to the real spatially repetitive one, would give the same "large scale" fields, that is to say, the real fields deprived of their high spatial frequency components. This is often satisfactory in practice, since measurements "at a point" do not reveal the exact field values there, only their spatial weighted averages over some region around the point. The constitutive parameters of the homogenized materials are obtained by solving the Maxwell equations with the real ϵ , σ , μ , but *on the cell C only*, and with appropriate boundary conditions. This auxiliary problem is called the "cell problem".

Metamaterials, however, are a challenge for this theory, because of the special role played by the *frequency* of the excitation in that case. Emergent properties of metamaterials (ϵ_{eff} and μ_{eff} with *negative* real part, which results in negative refraction effects) only hold in a narrow frequency band. The cell problem, whose solution yields the effective coefficients, must therefore be parameterized by frequency, which is not the case in the classical theory. A new approach to homogenization is thus required. This will be detailed in the full paper.

2. A frequency-dependent homogenization technique

For this summary, let's just give an overview. The cell C is a parallelepiped subtended by three independent vectors v_1, v_2, v_3 , such that constitutive parameters be "C-periodic": $\epsilon(x + v_i) = \epsilon(x)$ for all x and $i = 1, 2, 3$, and the same for other parameters. A part of space (comprised of a large number of translates of C) occupied by a material with such translation invariance. Before homogenization, the problem is $-\text{rot} \epsilon \text{rot} h + \text{rot} h = 0$ and $i\omega\mu h + \text{rot} e = 0$, with constitutive laws $d = \epsilon e$ and $b = \mu h$, plus boundary conditions (left unspecified here) which one assumes take charge of the source of the fields (current forced in and out, or else, tangential part of the incoming field at

the boundary of the device). Coefficients ϵ and μ can be complex (one merges σ and ϵ in the standard manner, $\epsilon = \epsilon_0(\epsilon'_r - i \epsilon''_r)$, where $\epsilon''_r = \sigma/\omega\epsilon_0$) and ω has a definite value.

One expects (as experiments strongly suggest...) the homogenized coefficients μ_{eff} and ϵ_{eff} to depend on ω , so ω should *explicitly* be present in the statement of the cell problem. Indeed, the cell problem is as follows. First, set

$$\mathbf{e} = \mathbf{e}_C - i\omega/2 \mathbf{B} \times \mathbf{x}, \quad \mathbf{h} = \mathbf{h}_C + i\omega/2 \mathbf{D} \times \mathbf{x}, \quad (1)$$

where \mathbf{e}_C and \mathbf{h}_C denote complex- and vector-valued "C-periodic" fields, that is to say, such that their tangential parts at points of the boundary of C that correspond by translation $\mathbf{v}_1, \mathbf{v}_2$, or \mathbf{v}_3 , be the same. There, \mathbf{B} and \mathbf{D} are two given (vector valued) *parameters*, corresponding to the large-scale averages of \mathbf{b} and \mathbf{d} . The notation $\mathbf{B} \times \mathbf{x}$ means, the *field* whose value of at point \mathbf{x} is $\mathbf{B} \times \mathbf{x}$, where \mathbf{x} is the *vector* from the origin to *point* \mathbf{x} . (The final result will not depend on the location of the origin, which will be taken, for convenience, at the center of the cell.) Observe that fields in (1) are the superposition, on cell C , of large-scale fields $\mathbf{E} - i\omega/2 \mathbf{B} \times \mathbf{x}$ and $\mathbf{H} + i\omega/2 \mathbf{D} \times \mathbf{x}$, which do satisfy the Maxwell equations, and of C-periodic corrections $\mathbf{e}_C - \mathbf{E}$ and $\mathbf{h}_C - \mathbf{H}$, where \mathbf{E} and \mathbf{H} are constant vectors. (The aim is to find these in terms of \mathbf{B} and \mathbf{D} . Heuristic considerations based on "energetic equivalence" of the original material and its homogenized counterpart will suggest how to do that.)

Next, solve Maxwell's equations, $-i\omega\epsilon\mathbf{e} + \text{rot } \mathbf{h} = 0$ and $i\omega\mu\mathbf{h} + \text{rot } \mathbf{e} = 0$, in C . (A proper weak formulation to this effect, as well as a numerical technique based on edge elements, will be discussed.) Last, compute the "power content" of the cell,

$$Z(\mathbf{B}, \mathbf{D}) = \int_C \mu \mathbf{h} \cdot \bar{\mathbf{h}} - \int_C \bar{\epsilon} \mathbf{e} \cdot \mathbf{e} \quad (2)$$

where the bar denotes conjugation. Since this is a quadratic function of \mathbf{D} and \mathbf{B} , one can find two vectors \mathbf{E} and \mathbf{H} , linear functions of \mathbf{D} and \mathbf{B} , such that $Z(\mathbf{B}, \mathbf{D}) \equiv \text{vol}(C) [\mathbf{B} \cdot \bar{\mathbf{H}} - \bar{\mathbf{D}} \cdot \mathbf{E}]$, so that \mathbf{E} and \mathbf{H} appear as the large-scale electric and magnetic fields. Hence a relation $\{\mathbf{D}, \mathbf{B}\} = \mathbf{M}(\omega) \{\mathbf{E}, \mathbf{H}\}$, characteristic of a fictitious homogenized material which behaves, *as far as large-scale local averages of active and reactive power are concerned*, like the original metamaterial. Matrix \mathbf{M} , of size 6×6 , is composed of four 3×3 blocks, the diagonal blocks being the expected tensors ϵ_{eff} and μ_{eff} (non-isotropic, as a rule), and the two off-diagonal blocks, null only in the case when (one of the translates of) the cell C has a center of symmetry, express the *chirality* of the equivalent homogeneous medium.

3. Emergent properties: Negative index

The crucial presence of a minus sign in the Lagrangian (2) is the reason why matrices ϵ_{eff} and μ_{eff} can have – thanks to internal resonance phenomena – eigenvalues with *negative* real part, as expected in interesting configurations, such as those with split rings. In fact, the interest of the method proposed here (validated by numerical experiments [2]), will be to search, by numerical simulations, varying the parameters and optimizing shapes, for configurations of this kind.

References

- [1] A. Bensoussan, J.L. Lions, and G. Papanicolaou, *Asymptotic methods for periodic structures*, North Holland, Amsterdam, 1978.
- [2] M. El Feddi, Z. Ren, A. Razek, A. Bossavit: "Homogenization Technique for Maxwell Equations in Periodic Structures", *IEEE Trans. on Magn.*, 33, 2 (1997) 1382-5.